Method of Analytical Regularization in Computational Electromagnetics

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SCIENCE IN UKRAINE, 2007

Fundamental science

University science

Applied & defense science

National Academy of Sciences

University R&D

Labs of x-USSR Ministries

Migration of labs in 1990s

Int’l projects: INTAS, STCU, CRDF, RS, CNRS,..

Funds

State budget: $20 b
For science: 0.3%

Co.= R&D Funds
Institute of Radiophysics and Electronics NASU since 1955

1991
900 staff: 250 scientists and engineers
Average age: 40
Budget: $100M eqv.
20% from NASU
80% from defense projects

R&D into mm and sub-mm waves

2007
700 staff: 220 scientists and engineers
Average age: 59
Budget: $2M
80% from NASU
20% from int’l projects

18 R&D Depts. + General Depts.

Director
Scientific Council elected for 4 years
Department of Computational Electromagnetics
since 1980

Group 1: Modeling of waveguide components, circuits, and sub-systems

Group 2: Modeling of antennas and optoelectronic components

http://www.ire.kharkov.ua/dep12
1. Fundamental Ideas of Analytical Regularization

Reduction of EM BVP to a Fredholm’s operator equation

Boundary-value problem

with unique solution:

- Maxwell equations
- Boundary conditions
- Edge condition
- Radiation condition

Integral equation
(or series equations)

\[ G X = Y \]

Suppose that:

\[ G = G_1 + G_2 \]

\[ A = G_1^{-1} G_2, \quad B = G_1^{-1} Y \]

\[ \| A \|_{L_2} < \infty, \quad B \in L_2 \]

Fredholm 2-nd kind IE or infinite-matrix equation

\[ G_1 \] is more singular than \[ G_2 \]

and \[ G_1^{-1} \] is known
2. Consequences of Analytical Regularization (a.k.a. Semi-Inversion)

Fredholm’s theorems for \( X + A(k)X = B \)

Existence of exact solution:

Follows from the Fredholm’s alternative: either unique solution exists or \( k \) is a point of the spectrum of natural modes

\[
X = (I + A(k))^{-1}B
\]

Stability of the matrix condition number:

\[
\text{cond}(I + A^N) \xrightarrow{N \to \infty} \text{cond}(I + A)
\]

\[
= \| I + A \| \cdot \| (I + A)^{-1} \| < \infty
\]

Point-wise convergence of discrete solutions:

\[
e(N) = \| X - X^N \| (\| X \|)^{-1}
\]

\[
\leq \| (I + A)^{-1} \| \cdot \| A - A^N \| \xrightarrow{N \to \infty} 0
\]
What is invertible?

\[ G = G_1 + G_2 \]

- **Canonical-shape part (circular cylinder, sphere, PEC strip)**
  - well developed – trigonometric basis; e.g., multiple canonical scatterers in free space & layered host medium
- **Zero-contrast part for dielectric cylinders and particles**
  - well developed – Muller equations, (loaded) volume IE
- **Static part for PEC and imperfect zero-thickness screens**
  - well developed – EFIE + Chebyshev basis in 2-D;
    variants – in FT domain, RHP (in periodic case)
- **High-frequency part (halfplane) for PEC and imperfect screens**
  - scarcely developed – most promising for solving quasioptical-size scatterers with economic algorithms
4. Dielectric Cylinder Scattering - Formulation

Boundary-value problem

Scattering by an arbitrary smooth dielectric cylinder. Incident field can be a plane wave or a directive beam emitted by a localized source.

wavenumber \( k_j = k \sqrt{\varepsilon_j \mu_j} \) \( \alpha_j = 1/\mu_j \) or \( 1/\varepsilon_j \)

2D BVP: Find \( u_j(\vec{r}), \ r \in D_j \ j=1,2, \) such that

1. Helmholtz equation off \( S \): \((\Delta + k_j^2)u_j^{sc}(\vec{r}) = 0\)

2. Boundary conditions at \( S \):

\[ u_1(\vec{r})|_S = u_2(\vec{r})|_S \quad \text{and} \quad \alpha_1 \frac{\partial u_1(\vec{r})}{\partial n}|_S = \alpha_2 \frac{\partial u_2(\vec{r})}{\partial n}|_S \]

3. Sommerfeld radiation condition
5. Muller’s Boundary Integral Equations

Small contrast inversion

Fields representation = combination of the single and double layer potentials

\[
\begin{align*}
    u_1(\vec{r}) &= \int_S \left[ p_1(\vec{r}_s) \frac{\partial G_1(\vec{r},\vec{r}_s)}{\partial \vec{n}_s} - q_1(\vec{r}_s) G_1(\vec{r},\vec{r}_s) \right] d\vec{l}_s & \quad \text{for } \vec{r} \in D_1, \\
    u_2(\vec{r}) &= \int_S \left[ q_2(\vec{r}_s) G_2(\vec{r},\vec{r}_s) - u_2(\vec{r}_s) \frac{\partial G_2(\vec{r},\vec{r}_s)}{\partial \vec{n}_s} \right] d\vec{l}_s + u_0(\vec{r}) & \quad \text{for } \vec{r} \in D_2.
\end{align*}
\]

Parameterization + Boundary conditions =>

**Uniquely solvable set of BIEs of the Fredholm 2\textsuperscript{nd} kind:**

\[
\begin{align*}
    p_1(t) - \int_0^{2\pi} p_1(t_s) A(t,t_s) \, dt_s + \int_0^{2\pi} q_1(t_s) B(t,t_s) \, dt_s &= L(t) u_0(t) \\
    \left(1 + \frac{\alpha_1}{\alpha_2}\right) \frac{q_1(t)}{2} - \int_0^{2\pi} p_1(t_s) C(t,t_s) \, dt_s + \int_0^{2\pi} q_1(t_s) D(t,t_s) \, dt_s &= L(t) \frac{\partial u_0(t)}{\partial \vec{n}}
\end{align*}
\]

\[\alpha_j = \mu_j \quad \text{or} \quad \varepsilon_j \quad \text{for } E- \quad \text{or} \quad H\text{-polarization}\]
6. Muller’s Boundary Integral Equations

Kernel properties & discretization

In one of the kernels, a \( \log \)-type singularity is kept; others are regular

\[
A(t, t_s) = L(t) \left( \frac{\partial G_1}{\partial n_s} - \frac{\partial G_2}{\partial n_s} \right) B(t, t_s) = L(t) \left( G_1 - \frac{\alpha_1}{\alpha_2} G_2 \right) \quad C(t, t_s) = L(t) \left( \frac{\partial^2 G_1}{\partial n_s \partial n} - \frac{\partial^2 G_2}{\partial n_s \partial n} \right)
\]

\[||A||, \ldots, ||D|| < (\varepsilon - 1) \text{Const} \]

i.e., zero-contrast part is inverted

MBIEs + trigonometric-Galerkin discretization

\[p(t)L(t) = \frac{2}{i\pi} \sum_{m=-\infty}^{\infty} p_n e^{imt} \]

If the natural parameterization is used: \( L(t) = 1 \)
7. MBIE Algorithm Properties

Test example

Homogeneous dielectric cylinder: «super-ellipse» = rectangle with smoothed edges:

\[(x/la)^{2\nu} + (y/a)^{2\nu} = 1\]

\[0 < \nu < \infty\]

Relative computational error, determined by norm in \(l_2\)

\[e(N) = \frac{\|Z^N - Z^{N+1}\|}{\|Z^N\|}\]

where

\[Z^N = \{p^n_N, q^n_N\}\]

3 digit accuracy is achieved if

\[N \approx kal\sqrt{\epsilon} + \nu + 10\]

"super-ellipse" cross-section

Computational error versus matrix block size \(N\)
8. Wave Focusing by 2-D Hemielliptic Lenses

Geometry & Principle

Cross-sectional contour: \( S = S_1 \cup S_2 \)

twice-continuous curve combined from smoothly joined halves of ellipse and super-ellipse

GO: Parallel rays come to the rear focus of the ellipse if the eccentricity is

\[ e = 1 / \sqrt{\varepsilon} \]
9. Wave Focusing by Hemielliptic Lenses

Field intensity at the flat side

Focusability vs $l_1$

$F_{x,y} = \frac{aI(x_m, y_m)}{|x, y_m - x, y_{0.5}|}$

Main focus x-coordinate vs $l_1$

max intensity vs $l_1$

$\varepsilon = 11.7$, $ka = 10$, $l_2 = 1.0457$, $v_1 = 10$, $v_2 = 1$, $\gamma = 0^0$, $N_{DF} = 2^9$
10. Wave Focusing by Hemielliptic Lenses

Internal Resonances in Extended Hemielliptic lens

Near-field intensity portraits for EHE and EHC silicon lenses for various values of extension parameter, \( l_1 \)

\[ \varepsilon = 11.7, \; k\alpha = 10, \; l_2 = 1.0457, \; v_1 = 10, \; v_2 = 1, \; \gamma = 0^0, \; N_{DF} = 2^{10}. \]
11. Natural Modes of Microcavities

Field patterns of natural modes in open 2-D dielectric resonators

Eigenfrequency boundary-value problem $\Rightarrow$ Muller’s boundary integral equations $\Rightarrow$ analytical extraction of circular-contour part $\Rightarrow$ determinant equation

$$\text{Det} \ (I+A(k))=0$$

Modes in a square cavity with rounded edges (super-ellipse with $\nu=10$)

and in a curved triangular cavity

Refractive index is 2.63
12. PEC Strip Scattering - Formulation

Boundary-value problem

Scattering by an arbitrary smooth open cylindrical PEC strip. Incident field is a plane wave in the RCS analysis and a directive localized feed field in the reflector antenna analysis

2-D BVP: find such \( u^{sc}(\vec{r}) \), \( u = u^{in} + u^{sc} \) that

1. Helmholtz equation off \( M \):

\[
(\Delta + k^2)u^{sc}(\vec{r}) = 0
\]

2. Boundary conditions at \( M \):

\[
\left. u(\vec{r}) \right|_{M} = 0, \quad E - \text{pol.}
\]

\[
\left. \frac{\partial u(\vec{r})}{\partial n} \right|_{M} = 0, \quad H - \text{pol.}
\]

3. Sommerfeld radiation condition
Singular Integral Equations

**Fields representation – single/double layer potential:**

\[
E\text{-pol.} \quad u(\vec{r}) = \int_M p(\vec{r}_s)G_0(\vec{r}, \vec{r}_s)dl_s, \quad H\text{-pol.} \quad u(\vec{r}) = \int_M q(\vec{r}_s)\frac{\partial}{\partial n_s}G_0(\vec{r}, \vec{r}_s)dl_s
\]

\[
G_0(\vec{r}, \vec{r}_s) = \frac{i}{4} H_0^{(1)}(k |\vec{r} - \vec{r}_s|) \quad \text{Free space Green's function}
\]

Parameterization of contour \( M \) => Boundary conditions => **Singular IEs of the 1\text{st} kind**

\[
E\text{-polarization} \quad \int_{-1}^1 p(t_s)G_0(t, t_s)L(t_s)dt_s = -u^{in}(t), \quad H\text{-polarization} \quad \frac{\partial}{\partial n} \frac{1}{-1} q(t_s)\frac{\partial}{\partial n_s}G_0(t, t_s)L(t_s)dt_s = -\frac{\partial u^{in}(t)}{\partial n}
\]

*Direct discretization of SIE does not guarantee convergence, is inefficient and inaccurate*
Static-part inversion: SIE conversion to a Fredholm 2-nd kind IE

**H-pol. IE for PEC strip can be integrated once to lead to an IE with a Cauchy singularity plus auxiliary relation**

\[ \tilde{q}(t) + \int_{-1}^{1} \tilde{q}(t_s) N(t, t_s) dt_s = -K_t f(t) \]

is a Fredholm IE of the 2-nd kind, where

\[ \tilde{q}(t) = q(t)L(t) \]

\[ N(t, t_s) = (1-t_s^2)^{-1/2} K_t h(t, t_s), \quad K_t f(t) = -\frac{i}{4\pi} \int_{-1}^{1} (1-x^2)^{1/2} f(x) \frac{dx}{x-t} \]

Singular operator is invertible - see Carleman 1928; Krein 1951; Erdogan&Gupta 1972

Numerical solution of a Fredholm 2-nd kind IE can be obtained by any not pathologic discretization scheme (collocation, Galerkin) leading to the Fredholm 2-nd kind matrices. Hence, solutions are stable and convergent

\[ A X = B \]
\[ X + CX = D \]
15. PEC Strip: Analytical Preconditioning

**Static-part inversion:** diagonalization with eigenfunctions

Green's function decomposition = **static singular** part + regular part

\[
G_0 (\vec{r}, \vec{r}_s) = \frac{i}{4} \ln |\vec{r} - \vec{r}_s| + R(k |\vec{r} - \vec{r}_s|)
\]

**E-pol.**

\[
\frac{1}{\ln(1- t_s)} \int_{-1}^{1} \frac{T_n(t_s)}{(1-t_s)^{1/2}} \ln(t-t_s)dt_s = \sigma_n T_n(t), \quad T_n(t) = \text{Chebyshev polynomials of the 1-st kind}
\]

**H-pol.**

\[
\frac{1}{\ln(1-t_s)} \int_{-1}^{1} \frac{U_n(t_s)}{(1-t_s)^{1/2}} \frac{\partial^2 \ln(|\vec{r} - \vec{r}_s|)}{\partial n^2} dt_s = \tau_n U_n(t), \quad U_n(t) = \text{Chebyshev polynomials of the 2-nd kind}
\]

To transform SIE to the Fredholm 2\textsuperscript{nd} kind matrix equation, take full set of the corresponding Chebyshev polynomials as a basis (i.e., make analytical preconditioning)

\[
A X = B
\]

\[
\begin{bmatrix}
\| & \| \\
\| & \| \\
\end{bmatrix}
\]

\[
X + CX = D
\]
Dependences of the condition number, and near-field and far-field errors on matrix size, $N$

RA: $d=10\lambda$, $f/d=0.5$, $kb=2.6$
Offset Parabolic Reflector Simulation

- The larger the reflector, the narrower the main beam width.
- $kb = 11$
- $d/\lambda = 47$
- $\beta = 140^0$

Phase: main beam is locally close to the plane wave
Near field perfectly illustrates the wave effects and the interference in near zone
17. Scattering by a Dielectric Strip Grating

Static-part inversion

Generalized boundary conditions on thin dielectric strips:

\[(E_T^+ + E_T^-) = 2R(H_T^+ - H_T^-), \quad (H_T^+ + H_T^-) = 2Q(E_T^+ - E_T^-)\]

where \[R = Q \zeta_r^2 = (i/2)\zeta_0 \zeta_r \cot(\sqrt{\varepsilon_r \mu_r \kappa_0 \tau / 2})\]

Fractions of transmitted, reflected and absorbed power versus normalized frequency, \(\varepsilon_r = 10+i, \ \varphi = 0^0, 2w/d=0.5, \ \sigma/d=0.01.\)

Fractions of transmitted, reflected and absorbed power versus normalized frequency, \(\varepsilon_r = 10+i, \ \varphi = 0^0, 2w/d=0.5, \ \sigma/d=0.01.\)
18. Discrete Luneburg Lens Fed by a Conformal Printed Feed

**Lens:**
concentric spherical layers of uniform dielectrics

**Patches:** PEC, zero-thickness, co-axial spherical disks, $0 \leq \theta \leq \pi$

\[ \varepsilon(r) = 2 - \left( \frac{r}{R} \right)^2 \]

Two conformal patches

- driving current: centered RED or TMD, as a probe or a slot model

\[
N_{\text{shell}} + 1 \quad N_{\text{shell}} \quad N_{\text{shell}}+1
\]
19. Discrete Luneburg Lens Fed by a Conformal Printed Feed

SCMA structure:

\[ \theta_{in} = 0.02^\circ, \]
\[ \theta_{out} = 0.04^\circ \]
\[ r_{in}/r_{out} = 0.999 \]

"large lens"

Lens structure:

\[ N_{shell} : \text{number of shells} \]
\[ \theta_i = 0, r_i = i/N_{shell} \]
\[ \epsilon_{ri} = 2 - [(2i-1)/2N_{shell}]^2 \]
\[ \forall i, 1 \leq i \leq N_{shell} \]

Lens => small shift of SCMA resonance + whispering-gallery modes
20. Conclusions: the Merits of Analytical Regularization and Preconditioning

- Generates convergent and economic scattering algorithms with easily controlled accuracy – no supercomputers
- Enables one to access quasioptical range
- Leads to reliable simulations predicting even finest resonance features
- Reduces eigenvalue problems to favorable determinant equations
- Is promising for iterative matrix solvers
- Can serve as a fast core for optimization
- Enables explicit asymptotic solutions